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Continual Lie algebras and noncommutative counterparts of exactly solvable models

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Abstract
Noncommutative counterparts of exactly solvable models are introduced on the basis of a generalization of Saveliev–Vershik continual Lie algebras. Examples of noncommutative Liouville and sin/*h*-Gordon equations are given. The simplest soliton solution to the noncommutative sine-Gordon equation is found.

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1. Introduction

Noncommutative field theories [12] attract attention due to their interesting internal structure and appearance in noncommutative geometry, string theory and other areas of theoretical physics. The first step in an investigation of integrable noncommutative theories is to consider analogues of solvable models on noncommutative *two-dimensional* manifolds. So far, various approaches such as the gauged bi-complex [15, 16] or the $*$-product zero curvature approach [14, 17] (see also [18]) were applied in attempts to construct noncommutative integrable counterparts of classical integrable models. At the same time, it was shown in [8] that a general field theory defined on a noncommutative spacetime is non-unitary. Nevertheless, in two-dimensional case one can consider noncommutative Euclidean models in order to avoid the non-unitarity.

The general idea is to introduce an associative $*$-product of functions depending on noncommutative coordinates. A natural candidate is the Moyal product [13]. Then one has to introduce a noncommutative counterpart of an integrable model system of equations and show that such a model is integrable. A first naive attempt is to substitute all multiplications of functions in a commutative integrable equation by a $*$-product. In the case of the Moyal deformation this method fails, in general, to preserve integrability. That is, in some cases, such as the $U(N)$ principal chiral model [14], the integrability survives, but not in most cases.

In this paper, we start with the definition of $*$-product Saveliev–Vershik continual Lie algebras, provide certain examples of mappings $(S_*, K_{*0,\pm})$ which determine such algebras, and then we propose a unifying approach to the construction of noncommutative counterparts of nonlinear exactly solvable models. Noncommutative models obtained in these frames appear to be generalizations of systems of integro-differential-type equations constructed on the base of Saveliev–Vershik continual Lie algebras. Taking into account the powerful experience of the group-theoretical (representation theory) approach to integrable models in the commutative case, we conjecture that noncommutative $*$-product continual Lie algebra models are also exactly solvable subject to certain conditions. In particular cases of the Liouville and sin/h-Gordon models, we show how to obtain their noncommutative analogues using $*$-product continual Lie algebras. In contrast to other approaches, we managed to avoid extra constraints on functions due to an appropriate choice of $A_{\pm}$-pairs constructed with the help of continual Lie algebra generators. Aiming at an investigation of more general representation theory solutions to noncommutative Toda models we present the simplest soliton solution to the noncommutative sine-Gordon equation.

2. Saveliev–Vershik continual Lie algebras

Continual Lie algebras were introduced in [1] and then studied in [2, 3, 5]. Suppose E is an arbitrary (noncommutative) associative algebra over a field $\mathcal{F}$ (for instance $\mathbb{R}$ or $\mathbb{C}$), and $K_0, K_{\pm} : E \times E \to E$ are bilinear mappings. The subspaces $\hat{g} = g_{-1} \oplus g_0 \oplus g_{+1}$ of the local part of a continual Lie algebra g are isomorphic (as vector spaces) to E. The subspace $g_i, i = 0, \pm 1$, consists of the elements $\{X_i(\phi), \phi \in E\}$, parametrized by elements of E (continuous set of roots). The generators $X_i(\phi)$ are subject to the defining commutation relations

$$[X_0(\phi), X_0(\psi)] = X_0([\phi, \psi]) \tag{2.1}$$

$$[X_0(\phi), X_{\pm 1}(\psi)] = X_{\pm 1}(K_{\pm}(\phi, \psi)) \tag{2.2}$$

$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(K_0(\phi, \psi)) \tag{2.3}$$

for all $\phi, \psi \in E$, where $[\phi, \psi] = \phi\psi - \psi\phi$. It is also assumed that Jacobi identities are satisfied which leads to the conditions on the mappings $K_{0,\pm}$

$$K_{\pm}([\phi, \psi], \chi) = K_{\pm}(\phi, K_{\pm}(\psi, \chi)) - K_{\pm}(\psi, K_{\pm}(\phi, \chi)) \tag{2.4}$$

$$[\psi, K_0(\phi, \chi)] = K_0(K_+(\psi, \phi), \chi)) + K_0(\phi, K_-(\psi, \chi)) \tag{2.5}$$

for all $\phi, \psi, \chi \in E$. Then an infinite-dimensional algebra $g(E; K) = g'(E; K)/J$ is called a *continual contragredient Lie algebra* with the local part $\hat{g}$ and the defining relations (2.1)–(2.3), where $g'(E; K)$ is a Lie algebra freely generated by $\hat{g}$, and J is the largest homogeneous ideal having trivial intersection with g_0 [2, 3].

When E is a *commutative* associative algebra (possibly without unity) over a field $\mathcal{F}$ and the mappings K_0, $K_+(\phi, \psi) = -K_-(\phi, \psi)$ have a *linear form of the action* on $E \times E$, (i.e., for example, $K_{\pm}(\phi, \psi) = \pm K\phi \cdot \psi$, $K_0(\phi, \psi) = K_s(\phi \cdot \psi)$, where $K, K_s : E \to E$), then conditions (2.4) and (2.5) following from Jacobi identities are satisfied automatically, and one arrives at the simplest class of Saveliev–Vershik continual Lie algebras. When the map K_s is an identity, then the map K is called the *Cartan operator* [2]. The definitions of *temperate, polynomial* (in the sense of Gelfand–Kirillov dimension), and *constant growth* of a continual Lie algebra are given in [2]. A continual Lie algebra $g(E; K)$ is called the algebra of *temperate growth* if for each subspace g_n there exists a finite-dimensional subspace $\mathcal{L}_n \subset g_1, \dim \mathcal{L}_n < \infty$, such that $g_n = [g_{n-1}, \mathcal{L}_n]$. The *constant growth* is determined by condition $g_n \simeq g_1 \simeq E$.

Let us enumerate principal examples of Saveliev–Vershik continual Lie algebras $g(E; K)$ [2, 3]:

(a) *Poisson bracket algebra* $g(E; K)$: E is an algebra of trigonometric polynomials on a circle, $K = S = -\mathrm{i}\partial/\partial z$, with the commutation relation $[X_n(\phi), X_m(\psi)] = \mathrm{i}X_{n+m}(m\phi'\psi - n\phi\psi')$, $g_n \simeq E$. It was shown in [2] that this algebra is isomorphic to the algebra of functions on a two-dimensional torus T^2 endowed with the standard Poisson bracket.
(b) *The simplest continual limit of* A_r: $g(E; \partial^2/\partial z^2)$.
(c) *Current algebra on a manifold* $g(E; k \otimes I) \simeq C^\infty(\mathcal{M}, g(k))$: E is a space of vector functions on a manifold $\mathcal{M}$, $g(k)$ is a simple Lie algebra with a Cartan matrix $\mathcal{K}$.
(d) *Algebras of diffeomorphisms* $g(E; I - T, I - T^{-1}) \simeq g(E; 2I - T - T^{-1})$: $E = C^\infty(\mathcal{M})$, $\mathcal{M}$ is a C-class manifold, T is C-diffeomorphism of $\mathcal{M}$, and $[X_n(\phi), X_m(\psi)] = X_{n+m}(\phi T^n\psi - n\psi T^m\phi')$.
(e) *Contact Lie algebras* [4].
(f) *Cartan–Hilbert operator Lie algebra* $g(E; I \pm \mathrm{i}H)$: E is a space of functions on $\mathbb{C}^1$ satisfying Hölder condition, and H is Hilbert transform.
(g) *Cross-product Lie algebras* $g(E; I - T, I - T^{-1}) \simeq g(2I - T - T^{-1})$ of finite sums $\{\sum_n \phi_n \otimes W^n\}$ with commutation relation $[\phi \otimes W^n, \psi \otimes W^m] = (\phi T^n\psi - \psi T^m\phi) \otimes W^{n+m}$, $\phi, \psi \in E$. These algebras include *Kac–Moody algebras*, *Lie algebras, associated with a circle rotation*, the algebra of *infinitesimal area-preserving diffeomorphisms* of a torus, *vector fields on a manifold* $\mathcal{M}$ ($g(E; V, V)$, $[X_n(\phi), X_m(\psi)] = X_{n+m}(m\phi V\psi - n\psi V\phi)$, where E is a $C^\infty(\mathcal{M})$ algebra, and V is a vector field on $\mathcal{M}$), *Fairlie–Fletcher–Zachos sine algebras* [6, 7], and cross-product Lie algebras with the Cartan operator defined on a *noncommutative associative algebra* [3].

Kac–Moody Lie algebras represent discrete limits of continual Lie algebras when $E = \mathbb{C}^n$ (with coordinate multiplication in some basis), and the Cartan operator is a generalized $n \times n$ Cartan matrix $\mathcal{K}$, while commutation relations on a standard set of generators have the usual form [11]. Note that the consideration of the quotienting $g(E; K) = g'(E; K)/J$ in the definition of a continual Lie algebra is equivalent in a Kac–Moody Lie algebra case to imposing the Serre conditions. More examples of Saveliev–Vershik continual Lie algebras can be found in [2, 3].

2.1. $*$-product continual Lie algebras

Suppose a *noncommutative* algebra E is endowed with an associative (in general noncommutative) $*$-product. Introduce a generalization of Saveliev–Vershik continual Lie algebras which would fit our purposes. Let S_* be a bilinear mapping $S_* : E \times E \to E$. We are interested in continual Lie algebras whose mappings $K_{0,\pm} : E \times E \to E$ act on the product $E \times E$ in such a way that functions of E are multiplied with the help of a chosen $*$-product. Denote such mappings by K_{*0} and $K_{*\pm}$.

We define a *$*$-product continual Lie algebra* $g(E; S_*, K_*) = g'(E; S_*, K_*)/J$ (where $g'(E; S_*, K_*)$ is generated by a local part $\hat{g} = g_{-1} \oplus g_0 \oplus g_{+1}$) with commutation relations on the elements of the subspaces $g_i = \{X_i(\phi), \phi \in E\}$, $i = 0, \pm 1$

$$[X_0(\phi), X_0(\psi)] = X_0(S_*(\phi, \psi)) \tag{2.6}$$

$$[X_0(\phi), X_{\pm 1}(\psi)] = X_{\pm 1}(K_{*\pm}(\phi, \psi)) \tag{2.7}$$

$$[X_{+1}(\phi), X_{-1}(\psi)] = X_0(K_{*0}(\phi, \psi)) \tag{2.8}$$

for all $\phi, \psi \in E$. Jacobi identities imposed on the generators $X_{0,\pm 1}$ imply the following conditions (which generalize (2.4) and (2.5)) on the mappings $(S_*, K_{*0,\pm})$:

$$K_{*\pm}(S_*(\phi, \psi), \chi) = K_{*\pm}(\phi, K_{*\pm}(\psi, \chi)) - K_{*\pm}(\psi, K_{*\pm}(\phi, \chi)) \tag{2.9}$$

$$S_*(\psi, K_{*0}(\phi, \chi)) = K_{*0}(K_{*+}(\psi, \phi), \chi) + K_{*0}(\phi, K_{*-}(\psi, \chi)) \tag{2.10}$$

$$S_*(\phi, S_*(\psi, \chi)) + S_*(\psi, S_*(\chi, \phi)) + S_*(\chi, S_*(\phi, \psi)) = 0 \tag{2.11}$$

for all $\phi, \psi, \chi \in E$. The anti-symmetry condition for (2.6) implies for any $\phi, \psi \in E$ that

$$S_*(\phi, \psi) = -S_*(\psi, \phi). \tag{2.12}$$

Algebras $g(E; S_*, K_*)$ as well as algebras $g(E; K)$ [2] are $\mathbb{Z}$-graded, i.e., $g = g(E; S_*, K_*)$ can be decomposed into a direct sum $g = \bigoplus_{n\in\mathbb{Z}} g_n$ of subspaces subject to the grading condition $[g_n, g_m] \subset g_{n+m}$. A nontrivial task is to find examples of such mappings $(S_*, K_{*0,\pm})$ that would satisfy conditions (2.9)–(2.12). Though some examples (with $S_*(\phi, \psi) \equiv 0$ or $S_*(\phi, \psi) = [\phi, \psi]$ with usual multiplication of functions) that involve noncommutative product have been found in [2, 3], we believe that the set of such mappings is much more broader and should include noncommutative generalizations of Saveliev–Vershik continual Lie algebra examples given in the previous subsection.

Let us present new *noncommutative* examples of mappings (S_*, K_*) that generalize *linear action mappings* on a *commutative* algebra E used in [2, 3]. We will take advantage of some of these $*$-product continual Lie algebra examples in the construction of noncommutative counterparts of classical exactly solvable models.

I. The simplest nontrivial example of mappings (S_*, K_*) is

$$S_*(\phi, \psi) = [\phi, \psi]_* \tag{2.13}$$

$$K_{*0}(\phi, \psi) = \phi * \psi \tag{2.14}$$

$$K_{*+}(\phi, \psi) = \phi * \psi \tag{2.15}$$

$$K_{*-}(\phi, \psi) = -\psi * \phi \tag{2.16}$$

where $[\phi, \psi]_* = \phi * \psi - \psi * \phi$. Another set of mappings (S_*, K_*) can be obtained interchanging K_{*+} with K_{*-}. The mappings in examples II–IV represent deformations of example I mappings.

II.

$$S_*(\phi, \psi) = K\phi * \psi - K\psi * \phi \tag{2.17}$$

$$K_{*0}(\phi, \psi) = K\phi * \psi \tag{2.18}$$

$$K_{*+}(\phi, \psi) = K\phi * \psi \tag{2.19}$$

$$K_{*-}(\phi, \psi) = -K\psi * \phi \tag{2.20}$$

$$K^2 = K \tag{2.21}$$

i.e., K is a projector. In examples II–IV we assume that a mapping K acts on a $*$-product of functions as $K(\phi * \psi) = K\phi * K\psi$. It is easy to note that example II can be obtained from example I by the transform $M_1(K) = m_* \circ K_1 \circ m_*^{-1}$, where m_* is a mapping $m_* : E \otimes E \to E * E$, m_*^{-1} is its inverse, and K_1 is the standard notation for $K \otimes Id$. Thus, $\left(S_*^{II}, K_{*0,\pm}^{II}\right) = M_1(K) \circ \left(S_*^{I}, K_{*0,\pm}^{I}\right)$.

III.

$$S_*(\phi, \psi) = [\phi, \psi]_* \tag{2.22}$$

$$K_{*0}(\phi, \psi) = \pm K\phi * \psi \tag{2.23}$$

$$K_{*+}(\phi, \psi) = K\phi * \psi \tag{2.24}$$

$$K_{*-}(\phi, \psi) = -\psi * \phi \tag{2.25}$$

$$K^2 = 1 \tag{2.26}$$

and $\left(K^{III}_{*0}, K^{III}_{*+}\right) = M_1(K) \circ \left(\pm K^{I}_{*0}, K^{I}_{*+}\right)$, while the mappings S_* and K_*, remain the same. One can also find examples similar to examples II and III where mappings (S_*, K_*) act on the second multiplier in $E \times E$ by a map K subject to the same conditions on K.

IV.

$$S_*(\phi, \psi) = [\phi, \psi]_* \tag{2.27}$$

$$K_{*0}(\phi, \psi) = K_0\phi * K_0\psi \tag{2.28}$$

$$K_{*+}(\phi, \psi) = K_+\phi * \psi \tag{2.29}$$

$$K_{*-}(\phi, \psi) = -\phi * K_-\psi \tag{2.30}$$

$$K_0 \circ K_+ = K_0 \circ K_- = 1 \tag{2.31}$$

where $K_{0,\pm}$ are some mappings. This example is given by $K^{IV}_{*0} = K_0 \circ K^{I}_{*0}$. $K^{IV}_{*+,-} = M_{1,2}(K_{+,-}) \circ K^{I}_{*+,-}$, $M_2(K) = m_* \circ K_2 \circ m_*^{-1}$, $K_2 = Id \otimes K$.

One can verify that the sets of mappings in examples I–IV satisfy conditions (2.9)–(2.11) following from Jacobi identities as well as the anti-symmetry condition (2.12). These sets of mappings (with a linear form of the (S_*, K_*)-action on $E \times E$) represent a small and the simplest subset of $*$-product continual Lie algebras. We will show in sections 3 and 4 that with any example of $*$-product continual Lie algebra one can associate a system of nonlinear equations which would play the role of a noncommutative analogue of a commutative integrable model. The limited scope of this paper does not allow us to elaborate more on all examples of $*$-product continual Lie algebras though some of them are rather interesting in applications to noncommutative geometry. We should mention that both ordinary and $*$-product continual Lie algebras have not been studied well enough yet. The structure theory, as well as a representation theory for them is far from complete. As in an ordinary Lie algebra case [9], a representation theory (and, in particular, analogues of highest weight modules) plays the central role in a construction of exact solutions to systems of equations obtained on the base of continual Lie algebras.

3. Noncommutative counterparts of exactly solvable models

In commutative two-dimensional case, an exactly integrable system may be introduced [9] through the zero curvature condition on holomorphic and antiholomorphic elements of a flat connection in the trivial holomorphic principal fibre bundle over a manifold $\mathcal{M}$ endowed with commutative coordinates $z^{\pm}$. Suppose g is a (infinite-dimensional) Lie algebra endowed with a $\mathbb{Z}$-grading. Then the general form of holomorphic and antiholomorphic elements $A_{\pm}$ of a connection is the following:

$$A_{\pm} = e^0_{\pm} + \sum_{a=1}^{m_{\pm}} e^a_{\pm} \tag{3.1}$$

where $g_{m_{\pm}}$ is the highest (lowest) subspace in the grading decomposition of g used in $A_{\pm}$, $e^0_{\pm} = \sum_{i=1}^{n_0} u_{\pm i}\, x^0_i, n^0 = \dim g_0$; $e^a_{\pm} = \sum_{i=1}^{n^a_{\pm}} f^a_{\pm i}\, x^a_{\pm i}$, $n^a_{\pm} = \dim g_{\pm a}$, $x^0_i \in g_0, x^a_{\pm i} \in g_{\pm a}$ are elements of the subspaces $g_{0,\pm a}$ of a chosen $\mathbb{Z}$-grading, and $u_{\pm i}(z^+, z^-)$, $f_{\pm i}(z^+, z^-)$ are *arbitrary* differentiable functions of $z^{\pm}$. The zero curvature condition $[\partial_+ + A_+, \partial_- + A_-] = 0$, leads to the system of equations

$$\partial_+ e^0_- - \partial_- e^0_+ + \left[e^0_+, e^0_-\right] + \sum_{a=1}^{\min(m_+,m_-)} \left[e^a_+, e^a_-\right] = 0 \tag{3.2}$$

$$\sum_{a=1}^{m_\pm} \left(\partial_\mp e^a_\pm + \left[e^0_\mp, e^a_\pm\right]\right) - \sum_{1\leqslant b<a\leqslant m_\pm} \left[e^a_\mp, e^b_\pm\right] = 0. \tag{3.3}$$

It turns out that the system (3.2), (3.3) is exactly integrable. The exact integrability of a model means that it is possible to find solutions depending on a sufficient number of arbitrary functions of $z^{\pm}$ in the form of a series which is finite in the case of a finite-dimensional Lie algebra, and infinite (though absolutely convergent) in a finite growth infinite-dimensional Lie algebra g case [9]. Exactly solvable models defined on multidimensional manifolds were introduced in [10].

On the base of $*$-product continual Lie algebras defined in the previous section, it is possible to construct noncommutative counterparts of exactly integrable models considered in [1–3]. Suppose E is a noncommutative associative algebra of functions depending on noncommutative variables $z^{\pm} = \left(z^{\pm}_\mu\right)$, which satisfy the commutation relations $\left[z^+_\mu, z^-_\nu\right] = \mathrm{i}\theta^{\mu\nu}$, where $\mu, \nu = 1, \ldots, N$, and $\theta^{\mu\nu}$ are real constants. In this paper we discuss models defined on a noncommutative *two-dimensional* manifold $\mathcal{M}$. Further generalizations to noncommutative multidimensional case will be presented in a separate paper. Introduce an associative and, in general, noncommutative $*$-product on E. Consider the elements of the form

$$A_\pm = X_0(u_\pm) + \sum_{a=\pm 1}^{m_\pm} \sum_{i=1}^{n^a_\pm} X^a_{\pm i}\left(f^a_{\pm i}\right) \tag{3.4}$$

where $X_0 \in g_0, X^a_{\pm i} \in g_{\pm a}, i = 1, \ldots, n^a_\pm = \dim g_{\pm a}$, and $u_{\pm_i}(z^+, z^-), f^a_{\pm_i}(z^+, z^-)$ are arbitrary differentiable functions of noncommutative coordinates $z^{\pm}$. Then the zero curvature condition

$$[\partial_+ + A_+, \partial_- + A_-] = 0 \tag{3.5}$$

applied to elements (3.4) generates the system of nonlinear equations

$$\partial_+ X_0(u_-) - \partial_- X_0(u_+) + X_0(S_*(u_+, u_-)) + \sum_{a=1}^{\min(m_+,m_-)} \left[e^a_+, e^a_-\right] = 0 \tag{3.6}$$

$$\sum_{a=1}^{m_\pm} \left(\partial_\mp e^a_\pm + \left[X_0(u_\mp), e^a_\pm\right]\right) - \sum_{1\leqslant b<a\leqslant m_\pm} \left[e^a_\mp, e^b_\pm\right] = 0 \tag{3.7}$$

where $e^a_\pm = \sum_{i=1}^{n^a_\pm} X^a_{\pm i}\left(f^a_{\pm i}\right)$. The system (3.6), (3.7) generalizes (3.2) , (3.3) as well as the system of equations introduced in [2] for the case of Saveliev–Vershik continual Lie algebras. Note that (3.6), (3.7) is much more nontrivial with respect to (3.2), (3.3) since it allows generation of noncommutative nonlinear systems of *integro-differential* equations. In this paper we consider the simplest case of elements (3.4) when $m_- = m_+ = 1$. The higher grading case, i.e., when $m_\pm > 1$, will be described elsewhere. Further in the paper a general noncommutative associative $*$-product is assumed unless we specify it precisely. One can take for instance the Moyal product [13]

$$f(z^+, z^-) * g(z^+, z^-) = m \circ e^P[f(z^+, z^-) \otimes g(\xi^+, \xi^-)]|_{\xi^\pm = z^\pm} \tag{3.8}$$

where $P = 1/2\theta^{\mu\nu}\partial_{+\mu}^{(z)} \otimes \partial_{-\nu}^{(\xi)}$, $\mu, \nu = 1, \ldots, N$, and m is the mapping $m : f \otimes g \to f \cdot g$, where the dot denotes usual multiplication of functions. Finally, we formulate the following:

Conjecture. *The system of equations (3.6) and (3.7) on functions of noncommutative variables $z^{\pm}$ associated with the Moyal product continual Lie algebra of the constant growth ($g_n \simeq g_1 \simeq E$) is exactly solvable. Its solutions depend on a set of functions of noncommutative variables subject to certain conditions.*

The proof of this statement is not complete at the moment though we plan to present it in a separate paper [21]. The main idea of the proof (i.e., the construction of exact solutions to the system (3.6), (3.7) subject to certain conditions on a $*$-product continual Lie algebra) lies in the usage (as in an ordinary Lie algebra case) of properties of highest weight module analogue of a $*$-product continual Lie algebra. Though we work in frames of the exact solvability approach, we would mention that (as it was shown in [14]) an associative $*$-product deformed zero curvature condition (3.5) considered as a compatibility condition for a pair of $*$-product deformed linear differential equations implies the existence of an infinite number of conserved charges (the invertibility of certain operators was assumed). This may serve as an indirect argument in support of our conjecture.

4. Examples

Exactly integrable models arising from Saveliev–Vershik continual Lie algebras were considered in [2, 3]. In particular, a class of equations constructed on the base of continual Lie algebras with a linear form of $K_{0,\pm}$-mappings action (for example, $K_{\pm}(\phi, \psi) = (K_{\pm}\phi) \cdot \psi$, $K_0(\phi, \psi) = K_0(\phi \cdot \psi)$) was discussed. We believe that $*$-product continual Lie algebras open the way to more sophisticated exactly solvable models. In this section we give examples of noncommutative models that correspond to continual Lie algebras with $*$-deformed mappings.

Noncommutative Liouville equation. Consider a $*$-product continual Lie algebra defined by (2.6–2.8) with the mappings (2.13–2.16) (example I). Let the elements $A_{\pm}$ be

$$A_+ = X_0(u) + X_+(1) \tag{4.1}$$

$$A_- = X_-(f) \tag{4.2}$$

where $u(z^+, z^-)$ and $f(z^+, z^-)$ are arbitrary functions depending on noncommutative coordinates $z^{\pm}$. The zero curvature condition (3.5) applied to (4.1), (4.2) gives

$$-\partial_-\big((f)^{-1}_{*L} * \partial_+ f\big) + f = 0. \tag{4.3}$$

We assume that there exists the left inverse function f_*^{-1} with respect to the $*$-product, i.e., $(f)^{-1}_{*L} * f = 1$. Now take $f = e_*^{\beta\phi}$, β is a constant (we assume also that the $*$-multiplication by a constant coincides with the usual multiplication) with the $*$-exponential defined by $e_*^x = \sum_{n=0}^{\infty} \frac{1}{n!} x_*^n$, where $x_*^n = x * \cdots * x$ (n-times $*$-product). Then we arrive at the noncommutative Liouville equation

$$\partial_-\big((e_*^{\beta\phi})^{-1}_{*L} * \partial_+ e_*^{\beta\phi}\big) = e_*^{\beta\phi}. \tag{4.4}$$

In the commutative limit, when the $*$-product is substituted with the usual product, we get the Liouville equation.

Noncommutative K-Liouville equation. Consider a $*$-product continual Lie algebra with the mappings (2.27)–(2.31) (example IV). The same $A_{\pm}$-pair (4.1), (4.2) subject to the curvature condition (3.5), leads to the model which we call the noncommutative K-Liouville equation

$$\partial_-\big(\partial_+ e_*^{\beta\phi} * (K_- e_*^{\beta\phi})^{-1}_{*R}\big) = K_0 e_*^{\beta\phi} \tag{4.5}$$

where the subscript R denotes the right inverse function with respect to the $*$-product.

Noncommutative sin/h-Gordon equation. Consider the pair of operators

$$A_+ = X_0(u) + X_+(1/2) + X_-(1/2) \tag{4.6}$$

$$A_- = X_-(f) + X_+\left(f^{-1}_{*L}\right) \tag{4.7}$$

where the generators $X_{0,\pm}(\phi)$ belong to a continual Lie algebra of the example I. Then the zero curvature condition (3.5) applied to (4.6), (4.7) leads to the equations

$$-\partial_-\big(\partial_+(e_*^{\mathrm{i}\beta\phi})^{-1}_{*L} * e_*^{\mathrm{i}\beta\phi}\big) = \tfrac{1}{2}\big(e_*^{\mathrm{i}\beta\phi} - (e_*^{\mathrm{i}\beta\phi})^{-1}_{*L}\big) \tag{4.8}$$

$$\partial_-\,\big((e_*^{\mathrm{i}\beta\phi})^{-1}_{*L} * \partial_+ e_*^{\mathrm{i}\beta\phi}\big) = \tfrac{1}{2}\big(e_*^{\mathrm{i}\beta\phi} - (e_*^{\mathrm{i}\beta\phi})^{-1}_{*L}\big) \tag{4.9}$$

where we have taken $f = e_*^{\mathrm{i}\beta\phi}$. We call (4.8), (4.9) the noncommutative counterpart of the sin/h-Gordon model. In a similar way, one can obtain the noncommutative Bullough–Dodd equation. Note that if the derivatives $\partial_\pm$ are with respect to the chosen $*$-product (i.e., the Leibnitz rule works) then $\partial_\pm\big((e_*^{\mathrm{i}\beta\phi})^{-1}_{*L} * e_*^{\mathrm{i}\beta\phi}\big) = 0$, and equations (4.8) and (4.9) are in fact equivalent. In the commutative limit, (4.8) and (4.9) collapse to one sin/h-Gordon equation.

Let a $*$-product in (4.8) and (4.9) be the Moyal product (3.8). Then (4.8) and (4.9) are equivalent to

$$\partial_-(e_*^{-\mathrm{i}\beta\phi} * \partial_+ e_*^{\mathrm{i}\beta\phi}) = \tfrac{1}{2}(e_*^{\mathrm{i}\beta\phi} - e_*^{-\mathrm{i}\beta\phi}). \tag{4.10}$$

It is easy to prove that the noncommutative sine-Gordon equation (4.10) possesses a soliton solution. Under a soliton solution to a noncommutative equation we mean a function of noncommutative coordinates $z^\pm$ which has a classical commutative soliton solution as the limit when $\theta = 0$. The soliton solution to (4.10) has the form

$$e_*^{-\mathrm{i}\beta\phi(z^+,z^-)} = (1 + e_*^\gamma) * (1 - e_*^\gamma)^{-1}_* \tag{4.11}$$

where $\gamma = z^- - z^+$. Here $(1-e_*^\gamma)^{-1}_*$ denotes the left inverse of the function $1-e_*^\gamma$ with respect to the Moyal product. As in the commutative limit, the noncommutative sine-Gordon equation possesses also a more general soliton solution $e_*^{-\mathrm{i}\beta\phi(z^+,z^-)} = (1 + Qe_*^\gamma) * (1 - Qe_*^\gamma)^{-1}_*$, where $\gamma = \omega_+ z^- - \omega_- z^+$, Q, $\omega_\pm$ are some constants, and $\frac{1}{2}$ in (4.10) is rescaled to $Q/2\omega_+\omega_-$.

The proof is rather direct. Note that due to the properties of the Moyal product and the linear dependence of γ on $z^\pm$, the noncommutative exponential e_*^γ coincides with the ordinary exponential e^γ. Also, the differentiations $\partial_\pm$ are with respect to the Moyal product and, in contrast to the case of $e_*^{\phi(z^+,z^-)}$, one can use the usual formulae when differentiating e^γ with respect to $z^\pm$. In order to prove (4.11) one has to substitute it into (4.10), and take into account the properties of e^γ. The solution (4.11) coincides with the classical soliton solution in the commutative limit. It can also be proved that a classical *multisoliton* solution to the sine-Gordon equation written in a form similar to (4.11) does satisfy the noncommutative sine-Gordon equation (4.10).

An N-soliton solution to the *commutative* sine-Gordon equation can be obtained from the general solution (found in [20] in frames of the group-representation approach [9])

$$\mathrm{e}^{-\mathrm{i}\beta\lambda\phi} = \frac{\langle\lambda_1|\gamma_+^{-1}\mu_+^{-1}\mu_-\gamma_-|\lambda_1\rangle}{\langle\lambda_0|\gamma_+^{-1}\mu_+^{-1}\mu_-\gamma_-|\lambda_0\rangle} \tag{4.12}$$

(where $\gamma_\pm, \mu_\pm$ are holomorphic and antiholomorphic group elements in the Gauss decomposition) through the ansatz

$$\mu_+^{-1}(0)\mu_-(0) = \prod_{i=1}^{N} Q_i\, e^{F_i(\zeta)} \tag{4.13}$$

(here ζ is related to the rapidity of a soliton) which contains elements of the vertex operator construction for the $\widehat{sl_2}$ algebra [11, 20] (see also [19]). Of course, the existence of a soliton solution to a noncommutative analogue of an integrable equation does not prove the integrability of such noncommutative counterpart. Nevertheless, preliminary calculations show that it is possible to rewrite the soliton solution (4.11) to (4.10) in a form which is very similar to (4.12) with ordinary vertex operators as well as highest weight vectors of the fundamental representation of the affine Lie algebra $\widehat{sl_2}$, though all expressions would contain the dependence on noncommutative coordinates. This fact is quite comprehensible, since the Lie algebra $\widehat{sl_2}$ represents a discrete limit of a $*$-product continual Lie algebra. Therefore, the representation theory of $\widehat{sl_2}$ should have some relations with a representation theory of corresponding continual Lie algebra. The construction of more general (soliton) solutions and proof of the exact integrability of the Liouville and sin/h-Gordon (and, in general, of conformal affine Toda models) noncommutative counterparts require further representation theory development for corresponding $*$-product continual Lie algebras. That point as well as a noncommutative vertex operator construction will be discussed in a subsequent paper [21].

4.1. Comparison to other approaches

In [14] some other version of the Moyal product, noncommutative sin/h-Gordon model was constructed. The main equation and two additional constraints have the form

$$\partial_+\partial_-\varphi + m^2/\beta\,\sinh_*(\beta\varphi) = 0 \tag{4.14}$$

$$\partial_-(e_*^{-\beta\varphi}) + \beta/2(e_*^{-\beta\varphi} * \partial_-\varphi + \partial_-\varphi * e_*^{-\beta\varphi}) = 0 \tag{4.15}$$

$$\partial_-(e_*^{\beta\varphi}) - \beta/2(e_*^{\beta\varphi} * \partial_-\varphi + \partial_-\varphi * e_*^{\beta\varphi}) = 0 \tag{4.16}$$

where m, β are constants, and $\sin_*(\beta\varphi) = \frac{1}{2\mathrm{i}}(e_*^{\mathrm{i}\beta\varphi} - e_*^{-\mathrm{i}\beta\varphi})$, $(\sinh_*(\beta\varphi) = \mathrm{i}\sin_*(\beta\varphi)|_{\beta\to-\mathrm{i}\beta})$. In [14] the system (4.15), (4.16) was obtained through the $*$-product zero curvature condition (3.5) applied to the $A_\pm$-elements constructed on the base of the Pauli matrices $\sigma_{\pm,3}, \sigma_\pm = 1/2(\sigma_x \pm \mathrm{i}\sigma_-)$ (i.e., a spin 1/2 matrix representation of the Lie algebra sl_2, which is a discrete limit of a continual Lie algebra), $A_+ = -m\lambda/2\big(e_*^{\beta\varphi}\sigma_- + e_*^{-\beta\varphi}\sigma_+\big)$, $A_- = m\lambda/2(\sigma_- + \sigma_+) - \beta/2\partial_-\varphi\sigma_3$. Using an expansion in the noncommutativity parameter θ, it was shown in [14] that a one-soliton solution to the classical sine-Gordon equation satisfies equations (4.15) and (4.16).

In [16] another noncommutative version of the Moyal product sine-Gordon equation was obtained through the bi-complex approach [15] to (noncommutative) integrable models. The main equation and the constraint are

$$\partial_-(e_*^{-\mathrm{i}\varphi} * \partial_+ e_*^{\mathrm{i}\varphi} - e_*^{\mathrm{i}\varphi} * \partial_+ e_*^{-\mathrm{i}\varphi}) = \mathrm{i}\gamma\,\sin_*\varphi \tag{4.17}$$

$$\partial_-(e_*^{\mathrm{i}\varphi} * \partial_+ e_*^{-\mathrm{i}\varphi} + e_*^{-\mathrm{i}\varphi} * \partial_+ e_*^{\mathrm{i}\varphi}) = 0 \tag{4.18}$$

where γ is a constant.

As it was mentioned in [14] that various noncommutative versions of an integrable model come from an ambiguity in a noncommutative counterpart of derivative containing expressions. We argue that all such versions can be obtained in the frames of our unifying

$*$-product continual Lie algebra approach. Note that a *matrix representation* for the sl_2 Lie algebra generators was used in order to construct the $A_\pm$-pair for the equations (4.14)–(4.16). The extra constraints come from the off-diagonal terms due to the choice of an *arbitrary* function equal to $\partial_-\varphi$. In contrast to that, in the example C we chose a suitable set of a continual Lie algebra mappings (S_*, K_*) in order to avoid any extra constraints. At the same time, it is possible to obtain the equations of noncommutative sin/h-Gordon systems constructed through the sl_2-matrix representation and the bi-complex approach by means of $A_\pm$ elements containing generators of certain degenerate $*$-product continual Lie algebras.

The authors of [14, 16] deal with noncommutative analogues of integrable equations in frames of complete integrability. In order to obtain noncommutative analogues of the sine-Gordon equation both approaches refer to a particular choice of matrices which represent the gauged bi-complex or a representation of sl_2 algebra. Though in [16] an infinite number of conserved quantities naturally appeared from the bi-complex construction, it is not known whether they are in involution. In contrast to that we are looking for exact solutions, and our approach (as well as that of [9] in an ordinary Lie algebra case) is more invariant since we do not use any kind of a representation in the elements $A_\pm$ which lead to nonlinear systems of equations.

5. Conclusions

We have introduced $*$-product continual Lie algebras which are generalizations of Saveliev–Vershik continual Lie algebras. New examples of mappings (S_*, K_*) that determine $*$-product continual Lie algebras are found. Taking advantage of such properties of Lie algebras we have formulated a new unifying approach to the construction of noncommutative counterparts of exactly solvable models. Then we conjecture the exact solvability of such counterparts. The noncommutative Liouville and $\sin/h$-Gordon equations are presented as examples, and the simplest soliton solution to the noncommutative sine-Gordon equation is found. There is no doubt that there exist more complicated (of nonlinear type mapping action) examples of $*$-product continual Lie algebras. Such examples should lead to noncommutative models of integro-differential type. Among the first aims is the development of a representation theory based procedure of solving nonlinear systems of equation introduced in this paper. Some progress has already been made [21] in the direction of noncommutative soliton vertex operators which, by analogy with the commutative case, should generate soliton solutions to noncommutative affine Toda models. On the other hand, it is possible to incorporate known examples of noncommutative analogues of completely integrable models [15, 17, 18] to the constructions discussed in the paper. Finally, we would mention the possibility for super-generalizations of continual Lie algebras, and construction of corresponding solvable models.

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